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by
ZVI SCHECHNER

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by

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ABSTRACT

An N-component parallel system is subjected to a known load program. As time passes, components fail in a random manner which depends on their individual load histories. At any time t, the surviving components share the total load according to some rule. The system's lifetime distribution is studied under various breakdown rules. Under the linear breakdown rule it is shown that if the load program is increasing the system lifetime is IFR. Using the notion of Schur convexity, stochastic comparison of different systems is obtained. It is also shown that the system failure time is asymptotically normally distributed as the number of components grows large. All these results hold under various load sharing rules, in fact, one can prove that the system lifetime distribution is invariant under different load sharing rules.

For a more general breakdown rule only the equal load sharing is considered. The asymptotic distribution of the system lifetime is shown to be normal. The asymptotic mean and variance are derived.

CHAPTER 1

INTRODUCTION AND SUMMARY

This report is concerned with the study of parallel systems whose components are subjected to a certain nonnegative time dependent load.

Consider a parallel system of n components. The system sustains a certain load (damage, wear) which varies in time, and thus each functioning component is exposed to a certain fraction of this load. As time passes, components fail in a random manner which depends on the amount of damage they have been exposed to. As the single components fail one by one, each of the surviving components is exposed to an increasing share of the total load. This model is known in the literature as the "load sharing model" and has been treated previously by Coleman (1958), Birnbaum-Saunders (1958) and more recently by Phoenix (1978). Coleman adapted this model in connection with the strength behavior of fiber bundles and his theoretical results were remarkably consistent with the experimental behavior of a variety of structural materials. Birnbaum and Saunders used the model to study the "Fatigue" of materials under conditions involving dynamic loading. They also reported a very good agreement with empirical data. Phoenix (1978) adapted Coleman's assumptions about the stochastic behavior of the single component under a given load program. He made use of Stigler (1974) paper to show that the system failure time is asymptotically normally distributed as the number of components grows large and thus generalized Coleman's results which dealt with the calculation of the asymptotic mean time to failure. All previous study assumes that all the components are identical, that is each individual component has the same stochastic behavior under a same load program. Furthermore, the surviving components at any time t are assumed to share the total load equally.

In this report we generalize these assumptions. We allow the components to be nonidentical and for some special case to share the total load in a fairly general way.

In Chapter 2 we adapt the linear breakdown rule that was essentially used by Birnbaum-Saunders. Loosely speaking, if a single component is subjected to a time dependent load program, the failure rate of the component at any time t is linear in the load at t. Birnbaum-Saunders were interested in certain statistical aspects of the system (e.g., estimating the number of components in the system and the deterioration factor of the individual component). We focus on the stochastic characteristics of the system. Among other things we prove that the components lifetimes are stochastically associated, that if the system's load program is nondecreasing, its failure time is IFR. Using the notion of Schur convexity, we obtain stochastic bounds on the system lifetime distribution. Asymptotic results are also available as the number of components in the system grows large. All these results are shown to hold under a fairly general load sharing rule. In fact, we prove that the distribution of time to system's failure is invariant under the load sharing rule.

In Chapter 3 we adapt Phoenix model and generalize it to the nonidentical component case. We obtain asymptotic results similar to those in Phoenix (1978) and also point out an unjustified statement in his paper and give our version for it.

CHAPTER 2

THE LINEAR LAW BREAKDOWN RULE

Consider a system of N components in parallel which are being subjected to a nonnegative time dependent load $L_s(\cdot)$. At any time t , all the surviving components share the load according to some rule. In the first three sections of this chapter, we assume that they share the load equally. This assumption is removed in Section 2.4, where we consider a more general rule. Thus for the time being, if i $(i < N)$ components have failed prior to time t , the actual load on each of the $N - i$ surviving components at t is $L_s(t)/N - i$.

Component i $(i = 1,2, \ldots, N)$ subjected to a known load program $l(\cdot)$ has a random failure time T_i . Throughout this chapter we assume that under $l(\cdot)$ the distribution of T_i is of the form:

$$F_i(t \mid l) = 1 - e^{-\beta_i \int_0^t l(u)du}$$

where β_i is a strictly positive constant which characterizes the durability of component i .

If we define

$$f_i(t \mid l) = \frac{d}{dt} F_i(t \mid l) \quad \text{and}$$

$$\lambda_i(t \mid l) = \frac{f_i(t \mid l)}{1 - F_i(t \mid l)} \qquad t \geq 0$$

then it is easy to check that $\lambda_i(t \mid l) = \beta_i l(t)$.

$\lambda_i(t \mid l)$ is the *failure rate function* of component i under the known load program $l(\cdot)$. It can be interpreted as the instantaneous probability of a t years old component i to fail, that is, component i having not failed prior to time t and sustaining load $l(t)$ will fail during the interval $(t, t + \Delta t)$ with probability $\beta_i l(t)\Delta t + o(\Delta t)$ where $o(\Delta t)$ denotes remainder terms of order less than Δt as $\Delta t \to 0$. For the obvious reason we say that the component obeys the linear law breakdown rule. As a system, the components lifetimes are, therefore, strongly dependent but only through their failure rate functions. For instance, if at t there are j $(1 < j \leq N)$ functioning components and a component fails, then under the equal load sharing rule the load that each of the surviving components sustain changes instantaneously from $L_s(t^-)/j$ at t^- to $L_s(t)/j - 1$ at t.

2.1 The Joint Distribution of Components Lifetimes

We denote by $T_1, T_2, \ldots, T_N$ the lifetimes of components $1,2, \ldots, N$ respectively and by $T_{(1)} \leq T_{(2)} \leq \cdots \leq T_{(N)}$ their order statistics. We assume that the system is being subjected to a non-negative load program $L_s(t)$ and that each component obeys linear law breakdown rule. Each of the components is being characterized by its proportional factor β (Birnbaum, Saunders (1958) called it the deterioration factor) which is assumed to be strictly positive number. The system itself is being characterized by its load program $L_s(\cdot)$ and by the vector

$$\underline{\beta} = (\beta_1, \beta_2, \ldots, \beta_N) .$$

The above assumptions lead to the following facts:

(2.1) For any i $(i = 1,2, \ldots, N)$,

$$P(T_i \in (t\ ,\ t + \Delta t] \mid T_{(1)} > t) = \beta_i \frac{L_s(t)}{N} \Delta t + o(\Delta t)$$

as $\Delta t \to 0$.

(2.2) If $0 \leq t_1 \leq t_2 \leq \cdots \leq t_k \leq t$, $k < j \leq N$, and $\{n_1, n_2, \ldots, n_N\}$ is an arbitrary permutation of $\{1,2, \ldots, N\}$, then:

$$P\left(T_{n_j} \in (t\ ,\ t + \Delta t] \mid T_{n_1} = t_1, \ldots, T_{n_k} = t_k\ ,\ T_{(k+1)} > t\right) =$$

$$\beta_{n_j} \frac{L_s(t)}{N - k} \Delta t + o(\Delta t)$$

as $\Delta t \to 0$.

Facts (2.1), (2.2) tell us that if exactly k $(k = 0, \ldots, N - 1)$ components failed by time t and component j is not among them, then the failure rate of j at t is

$$\beta_j \frac{L_s(t)}{N - k} .$$

From (2.2) we also learn that the order in which the components failed does not affect the failure rate of component j , as long as it is given that $n - j$ survived t and j is among them.

<u>Lemma 2.1</u>:

$$P(T_{(1)} > t) = e^{-\frac{\beta_1 + \cdots + \beta_N}{N} \int_0^t L_s(u)du} .$$

Proof:

From (2.1), it is obvious that the failure rate function of $T_{(1)}$ is

$$\frac{L_s(t)}{N} \sum_{j=1}^{N} \beta_j .$$

Q.E.D.

Lemma 2.2:

Let $0 < t_1 \leq t_2 \leq \ldots \leq t_k \leq t$, $k < N$ and let $\{n_1, n_2, \ldots, n_N\}$ be an arbitrary permutation of $\{1,2, \ldots, N\}$. Then for $x > 0$ we have

$$P\left(T_{(k+1)} > t + x \mid T_{n_1} = t_1, \ldots, T_{n_k} = t_k , T_{(k+1)} > t\right) =$$

$$\exp - \left\{\frac{\beta_{n_{k+1}} + \ldots + \beta_{n_N}}{N - k} \int_t^{t+x} L_s(u)du\right\} .$$

Proof:

Follows from (2.2) in a straightforward way.

Q.E.D.

Lemma 2.3:

$\{t_i\}$, $\{n_i\}$ as in Lemma 2.2 and let $\underline{dt}$ denote the infinitesimal interval $(t , t + dt)$, then for $x > 0$:

$$P\left(T_{n_k} \in \underline{dt} , T_{(k+1)} > t + x \mid T_{n_1} = t_1, \ldots, T_{n_{k-1}} = t_{k-1} , T_{(k)} > t\right) =$$

$$\left[\frac{\beta_{n_k}}{N - k + 1} L_s(t) \exp \left\{- \frac{\beta_{n_{k+1}} + \ldots + \beta_{n_N}}{N - k} \int_t^{t+x} L_s(u)du\right\}\right]dt .$$

Proof:

$$P\left(T_{n_k} \in \underline{dt} , T_{(k+1)} > t + x \mid T_{n_1} = t_1, \ldots, T_{n_{k-1}} = t_{k-1} , T_{(k)} > t\right) =$$

$$P\left(T_{(k+1)} > t + x \mid T_{n_1} = t_1, \ldots, T_{n_{k-1}} = t_{k-1} , T_{n_k} = t_{k-1} , T_{n_k} = t ,\right.$$

$$\left. T_{(k+1)} > t\right) \cdot P\left(T_{n_k} \in \underline{dt} \mid T_{n_1} = t_1, \ldots, T_{n_{k-1}} = t_{k-1} , T_{(k)} > t\right) .$$

Combine (2.2) and Lemma 2.2 and the result follows.

Q.E.D.

We are now ready to state the following theorem:

Theorem 2.1:

Consider the simplex $0 \le t_1 \le \ldots \le t_N$ and let $\{n_1, n_2, \ldots, n_N\}$ be a permutation of $\{1, 2, \ldots, N\}$, then

$$P\left(T_{n_1} \in \underline{dt}_1, T_{n_2} \in \underline{dt}_2, \ldots, T_{n_N} \in \underline{dt}_N\right) =$$

$$\frac{\prod_{j=1}^{N} \beta_j L_s(t_j) dt_j}{N!} \cdot \exp - \left[\frac{\beta_{n_1} + \ldots + \beta_{n_N}}{N} \int_0^{t_1} L_s(u) du\right.$$

$$\left. + \frac{\beta_{n_2} + \ldots + \beta_{n_N}}{N - 1} \int_{t_1}^{t_2} L_s(u) du + \ldots + \beta_{n_N} \int_{t_{N-1}}^{t_N} L_s(u) du\right] .$$

Proof:

$$P\left(T_{n_1} \in \underline{dt}_1, \ldots, T_{n_N} \in \underline{dt}_N\right) =$$

$$P\left(T_{n_N} \in \underline{dt}_N \mid T_1 \in \underline{dt}_1, \ldots, T_{n_{N-1}} \in \underline{dt}_{N-1}, T_{(N)} > t_N\right)$$

$$\times P\left(T_{n_{N-1}} \in \underline{dt}_{N-1}, T_{n_N} > t_N \mid t_{n_1} \in \underline{dt}_1, \ldots, T_{n_{N-2}} \in \underline{dt}_{N-2}, T_{(N-1)} > t_{N-1}\right)$$

$$\times P\left(T_{N_1} \in \underline{dt}_1, T_{(2)} > t_2 \mid T_{(1)} > t_1\right)$$

$$\times P(T_{(1)} > t_1) .$$

Combine Lemmas 2.1, 2.2, 2.3 and the result follows.

Q.E.D.

Corollary 2.1:

If $\beta_1 = \beta_2 = \ldots = \beta_N = \beta$, then the density function of $T_{(N)}$ is given by

$$f_{T_{(N)}}(t) = \beta^N L_s(t) \frac{\left(\int_0^t L_s(u)\,du\right)^{N-1}}{(N-1)!} e^{-\beta \int_0^t L_s(u)du} .$$

Proof:

By Theorem 2.1, we have that on $0 < t_1 < \ldots < t_N$

$$P(T_{(1)} \in \underline{dt}_1, \ldots, T_{(N)} \in \underline{dt}_N) = \left(\prod_{j=1}^{N} L_s(t_j)\beta dt_j\right) e^{-\beta \int_0^t L_s(u)du} .$$

Integration yields the result.

Q.E.D.

In fact, this result can be obtained directly from the simple observation that if the β's are equal then

$$P(T_{(i)} \in \underline{dt} \mid T_{(1)}, \ldots, T_{(i-1)}, T_{(i)} > t)$$

$$= \begin{cases} \beta L_s(t)dt & \text{on } T_{(i-1)} \leq t \\ 0 & \text{on } T_{(i-1)} > t . \end{cases}$$

Thus the ordered sequence of failure times $T_{(1)} \leq T_{(2)} \leq \cdots \leq T_{(N)}$ *behaves like a sequence of* "events" taken from a nonhomogeneous Poisson process with intensity function $\beta L_s(\cdot)$, and

$$P(T_{(N)} < t) = \Pr\{\text{there are at least } N \text{ events of the Poisson in } (0,t]\}$$

$$= 1 - \left[\sum_{j=0}^{N} \left(\beta \int_0^t L_s(u)\right)^j / j!\right] e^{-\beta \int_0^t L_s(u)du}$$

and $\frac{d}{dt} P(T_{(N)} \leq t)$ yields the result.

Let us now define the random process $L(t)$

$$(2.3) \qquad L(t) = \begin{cases} L_s(t)/N & 0 \leq t < T_{(1)} \\ \dfrac{L_s(t)}{N - i} & T_{(i)} \leq t < T_{(i+1)} \quad i = 1,2, \ldots, N-1 \\ L_s(t) & t > T_{(N)} \end{cases}$$

and let

$$(2.4) \qquad Q_i = \int_0^{T_i} L(u)du \ , \ i = 1,2, \ \ldots, \ N \ .$$

Thus intuitively, Q_i is the total load sustained by component i during its lifetime.

Theorem 2.2:

The random variables $Q_1, Q_2, \ldots, Q_N$ are independent exponentially distributed with rates $\beta_1, \beta_2, \ldots, \beta_N$ respectively.

Proof:

The joint density function of the Q's can be written in terms of the joint density of the T's :

$$f_{\underline{Q}}(q_1, q_2, \ldots, q_N) = f_{\underline{T}}(t_1, t_2, \ldots, t_N) ||J||^{-1}$$

where $t_i = t_i(q_1, \ldots, q_N)$, $i = 1,2, \ldots, N$ and $||J|| = \left|\left|\frac{\partial(q_1, \ldots, q_N)}{\partial(t_1, \ldots, t_N)}\right|\right|$ is the Jacobian of $\underline{q}$ with respect to $\underline{t}$.

Now consider the simples $0 < q_1 < q_2 < \ldots < q_N$ and let $t_1, t_2, \ldots, t_N$ be such that

$$q_1 = \int_0^{t_1} \frac{L_s(u)}{N}\, du$$

$$q_2 = q_1 + \int_{t_1}^{t_2} \frac{L_s(u)}{N - 1}\, du$$

$$\vdots$$

$$q_N = q_{N-1} + \int_{t_{N-1}}^{t_N} L_s(u)du \ .$$

Clearly $t_1 < t_2 < \ldots < t_N$ and the joint density function of $Q_1, \ldots, Q_N$ on the above simplex is

$$f_{\underline{Q}}(q_1, \ldots, q_N) = \left[\frac{\prod_{i=1} \beta_i L_s(t_i(q_1, \ldots, q_N))e^{-\beta_i q_i}}{N!} \right] ||J||^{-1} \ .$$

By simple calculus we also have

$$J = \begin{pmatrix} \frac{L_s(t_1)}{N} & & & \bigcirc \\ \vdots & \frac{L_s(t_2)}{N - 1} & & \\ \vdots & & \ddots & \\ \cdots & \cdots & \cdots & L_s(t_N) \end{pmatrix}$$

and hence

$$||J|| = \frac{\prod_{i=1}^{N} L_s(t_i)}{N!} .$$

Thus on $0 < q_1 < q_2 < \dots < q_N$, the density on the Q's is

$$f_{\underline{Q}}(q_1, \dots, q_N) = \prod_{j=1}^{N} \beta_j e^{-\beta_j q_j} .$$

It is fairly easy to check this argument holds for an arbitrary simplex of the form $0 < q_{n_1} < \dots < q_{n_N}$, where $\{n_1, n_2, \dots, n_N\}$ is a permutation of $\{1,2, \dots, N\}$. Thus the joint density function of $Q_1, Q_2, \dots, Q_N$ is given by

$$f_{Q_1, \dots, Q_N}(q_1, \dots, q_N) = \prod_{i=1}^{N} \beta_i e^{-\beta_i q_i} , \; q_i > 0 , \; i = 1,2, \dots, N .$$

Q.E.D.

The importance of this theorem is that it establishes the existence of a transformation ψ_{L_s} , $\psi_{L_s} : (T_1, \dots, T_N) \to (Q_1, \dots, Q_N)$ which under a given load program $L_s(\cdot)$ maps the random vector $\underline{T}$ with dependent components into $\underline{Q}$ with independent components. This fact enables us to analyze most of the parameters of the system in terms of independent random variables. Indeed, throughout the next two sections we use this fact to find conditions under which the system lifetime is IFR, obtain bounds for the system's reliability function as well as to establish the asymptotic distribution of the system's lifetime as N, the number of components, increases to infinity.

We start with proving the rather intuitive fact that the lifetimes of the individual components $(T_1, \ldots, T_N)$ are associated, but first some necessary preliminaries.

Definition:

We say that the random variables $Z_1, Z_2, \ldots, Z_n$ are associated if: $\mathrm{Cov}\,(f(\underline{Z}), g(\underline{Z})) \geq 0$ for all nondecreasing functions (f,g) for which the covariance is well defined. We use $\underline{Z}$ for $(Z_1, \ldots, Z_n)$.

For completeness we start with the following properties of association:

(P_1) Any subset of associated random variables are associated.

(P_2) If two sets of associated random variables are independent of one another, then their union is a set of associated random variables.

(P_3) The set consisting of a single random variable is associated.

(P_4) Nondecreasing functions of associated random variables are associated.

(P_5) Independent random variables are associated.

From the way $\psi_{L_s} : (T_1, \ldots, T_N) \to (Q_1, \ldots, Q_N)$ is constructed, it is clear that if the load program $L_s(\cdot)$ is strictly positive for each t, ψ_{L_s} is one to one and hence $\psi_{L_s}^{-1}$ is also well defined. Now in order to prove that $(T_1, T_2, \ldots, T_N)$ are indeed associated, it suffices to show that $\psi_{L_s}^{-1}$ is nondecreasing, that is, $T_1(Q_1, \ldots, Q_N), T_2(Q_1, \ldots, Q_N), \ldots, T_N(Q_1, \ldots, Q_N)$ are all non-decreasing.

Lemma 2.4:

If $L_s(\cdot) > 0$, then $\psi_{L_s}^{-1}$ is well defined and increasing functional.

Proof:

The first part follows easily from the way ψ_{L_s} is constructed. To show that $\psi_{L_s}^{-1} : (q_1, \ldots, q_N) \to (t_1, \ldots, t_N)$ is increasing functional, denote by $q_{(1)} \le q_{(2)} \le \cdots \le q_{(N)}$ and $t_{(1)} \le t_{(2)} \le \cdots \le t_{(N)}$ the increasing rearrangement of $q_1, \ldots, q_N$, $t_1, \ldots, t_N$ respectively. Under $\psi_{L_s}^{-1}$, $t_{(1)}(q_1, \ldots, q_N), t_{(2)}(q_1, \ldots, q_N), \ldots, t_{(N)}(q_1, \ldots, q_N)$ are such that

$$q_{(1)} = \int_0^{t_{(1)}} \frac{L_s(u)}{N}\, du$$

$$q_{(2)} = \int_0^{t_{(1)}} \frac{L_s(u)}{N}\, du + \int_{t_{(1)}}^{t_{(2)}} \frac{L_s(u)}{N-1}\, du$$

$$\vdots$$

$$q_{(N)} = \int_0^{t_{(1)}} \frac{L_s(u)}{N}\, du + \ldots + \int_{t_{N-1}}^{t_N} L_s(u)du\ .$$

Now fix all q's and increase $q_{(i)}$ by the infinitesimal increment $dq_{(i)}$. It is easy to check that for $j < i$, $t_{(j)}$ remains unchanged. For $t_{(i)}$ we have

$$dq_{(i)} = \frac{L_s(t_{(i)})}{N - i + 1} dt_{(i)} .$$

To find $dt_{(i+1)}$ note that

$$0 = dq_{(i+1)} = dq_{(i)} + \frac{L_s(t_{(i+1)})}{N - i} dt_{(i+1)} - \frac{L_s(t_{(i)})}{N - i} dt_{(i)}$$

or

$$dt_{(i+1)} = \frac{L_s(t_{(i)})}{L_s(t_{(i+1)})} \frac{dt_{(i)}}{N - i + 1}$$

and indeed for $k \geq 1$

$$dt_{(i+k)} = \frac{L_s(t_{(i)})}{L_s(t_{(i+k)})} \frac{dt_{(i)}}{N - i + 1} , \; k = 1,2, \ldots, N - i$$

which completes the proof.

Q.E.D.

Corollary 2.2:

The components lifetimes $T_1, \ldots, T_N$ are associated.

Next we show that under increasing load program the system lifetime is IFR.

Definition:

A nonnegative random variable X with distribution $F(\cdot)$ is said to be IFR (or equivalently is said to have an increasing failure rate distribution) if $\log(1 - F(x))$ is concave where finite.

If F also possess a density $f(\cdot)$, then it is easy to check that the above is equivalent to saying that $\lambda(t) = \frac{f(t)}{1 - F(t)}$ is non-decreasing on $\{t : F(t) < 1\}$.

Lemma 2.5:

Let g be an increasing differentiable function and let (X,Y) be two random variables such that $Y = g(X)$ and suppose Y has a failure rate function λ_Y, then the failure rate function of X is given by $\lambda_X(t) = \lambda_Y(g(t))g'(t)$.

Proof:

$$\lambda_X(t) = \frac{f_X(t)}{1 - F_X(t)} = \frac{f_Y(g(t))g'(t)}{1 - F_Y(g(t))} = \lambda_Y(g(t))g'(t) .$$

Q.E.D.

Theorem 2.3:

If the system load program $L_s(\cdot)$ is nondecreasing, then the system lifetime is IFR.

Proof:

Let $Q_1, Q_2, \ldots, Q_N$ be as defined in (2.4), then check that

$$\sum_{i=1}^{N} Q_i = \int_0^{T_{(N)}} L_s(u)du .$$

Now since the Q's are independent IFR, so is $\sum_{i=1}^{N} Q_i$ and by Lemma 2.5 we have

$$\lambda_{T_{(N)}}(t) = \lambda_{\Sigma Q}\left(\int_0^t L_s(u)\right) L_s(t) .$$

Hence $\lambda_{T_{(N)}}$ is nondecreasing.

Q.E.D.

2.2 Schur Functions and System Reliability

The method and concept of majorization and Schur functions are used in this section to make stochastic comparison between systems with different components (that is, components with different β-factor) and to develop conservative bounds for the system lifetimes. For the sake of completeness we review some of the definitions and tools of majorization and Schur functions:

Given a vector $\underline{x} = (x_1, \ldots, x_n)$ let $x_{[1]} \geq x_{[2]} \geq \cdots \geq x_{[n]}$ denote the decreasing rearrangement of $x_1, x_2, \ldots, x_n$.

Definition:

A vector $\underline{x}$ is said to majorize a vector $\underline{x}'$ if

$$\sum_{i=1}^{j} x_{[i]} \geq \sum_{i=1}^{j} x'_{[1]} \quad \text{for } j = 1, \ldots, n-1$$

and

$$\sum_{i=1}^{n} x_{[i]} = \sum_{i=1}^{n} x'_{[i]} .$$

We write $\underline{x} \overset{m}{\geq} \underline{x}'$.

Although majorization involves comparison of vectors of order n, the following characterization shows that we need only consider a pair of coordinates at a time.

Theorem 2.4: (Hardy, Littlewood, Polya, 1952, p. 47)

Let $\underline{x} \overset{m}{\geq} \underline{x}'$. Then $\underline{x}'$ can be obtained from $\underline{x}$ by finite number of T-transformations, where a T-transformation changes two coordinates only, and in the following way. If for example coordinates 1 and 2 are being transformed, then $T(\underline{y}) = (y_1', y_2', y_3', \ldots, y_n')$ where $\max\{y_1, y_2\} > \max\{y_1', y_2'\}$ and $y_1 + y_2 = y_1' + y_2'$. Furthermore we have $\underline{y} \overset{m}{\geq} T(\underline{y})$.

Majorization represents a partial ordering in $\mathbb{R}^n$. A Schur function is a function that is monotone with respect to this partial ordering, more formally:

Definition:

A function $h(\cdot)$ satisfying the property that $\underline{x} \overset{m}{\geq} \underline{x}'$ implies $h(\underline{x}) \geq (<)\ h(\underline{x}')$ is called *Schur convex* (Schur concave) function.

A useful characterization of Schur functions is provided by the fundamental theorem of Ostrowski (1964) which states:

Theorem 2.5: (Ostrowski)

A differentiable, permutation invariant function h on $\mathbb{R}^n$ is Schur convex (Schur concave) iff

$$(x_i - x_j)\left(\frac{\partial h}{\partial x_i} - \frac{\partial h}{\partial x_j}\right) \geq (\leq)\ 0 \quad \text{for all} \quad i \neq j\ .$$

We use this to prove the following:

Theorem 2.6:

If X_1 , X_2 are independent exponentially distributed random variables with rates λ_1 , λ_2 respectively and X_1' , X_2' are independent exponentially distributed with rates λ_1' , λ_2' respectively, and if $(\lambda_1,\lambda_2) \overset{m}{\geq} (\lambda_1',\lambda_2')$, then for any $t \geq 0$ $P(X_1 + X_2 \geq t) \geq P(X_1' + X_2' \geq t)$.

Proof:

By using the Ostrowski Theorem it is not difficult show that

$$P(X_1 + X_2 \geq t) = \frac{\lambda_1 e^{-\lambda_2 t} - \lambda_2 e^{-\lambda_1 t}}{\lambda_1 - \lambda_2}$$

is Schur convex in (λ_1,λ_2) .

Q.E.D.

Corollary 2.3:

If $X_1,X_2, \ldots, X_n$ and $Y_1,Y_2, \ldots, Y_n$ are two sets of independent exponential random variables with rates $(\lambda_1, \ldots, \lambda_n)$ and $(\lambda_1', \ldots, \lambda_n')$ respectively and if $\underline{\lambda} \overset{m}{\geq} \underline{\lambda}'$, then for any $t \geq 0$

$$P(X_1 + \ldots + X_n \geq t) \geq P(Y_1 + \ldots + Y_n \geq t) .$$

Proof:

Use Theorems 2.4, 2.6 and the well known fact that if X_1 , X_2 are independent and X_1' , X_2' are independent, and for any t , $P(X_i \geq t) \geq P(X_i' \geq t)$, $i = 1,2$, then for any t : $P(X_1 + X_2 \geq t)P(X_1' + X_2' \geq t)$.

Q.E.D.

Remark:

Proschan and Sethuraman (1974) proved Lemma 2.6 using a slightly different approach.

Corollary 2.4:

Fix $t > 0$, then

$$P\left(\int_0^{T_{(N)}} L_s(u)du > t \right)$$

is Schur convex function in $(\beta_1, \beta_2, \ldots, \beta_N)$.

Proof:

$$\int_0^{T_{(N)}} L_s(u)du = Q_1 + \ldots + Q_N$$

which are independent exponentially distributed with rates $\beta_1, \ldots, \beta_N$ respectively.

Q.E.D.

Corollary 2.5:

Fix $t > 0$, then $P(T_{(N)} > t)$ is Schur convex function in $\beta_1, \beta_2, \ldots, \beta_N$.

Proof:

Follows from Corollary 2.4

Q.E.D.

The following theorem provides a lower bound for the survival probability of a system under an arbitrary load program $L_s(\cdot)$.

Theorem 2.7:

Let

$$\beta = \frac{1}{N} \sum_{j=1}^{N} \beta_j \ ,$$

then for any $t > 0$,

$$P(T_{(N)} > t) \geq e^{-\beta \int_0^t L_s(u)du} \left[\sum_{j=0}^{N-1} \frac{\beta^j}{j!} \left(\int_0^t L_s(u)du \right)^j \right] .$$

Proof:

First note that $(\beta_1, \ldots, \beta_N) \overset{m}{\geq} (\beta, \beta, \ldots, \beta)$. The result follows from Corollaries 2.1, 2.5.

Q.E.D.

2.3 Asymptotic Distribution of System Lifetime

In this section we obtain the asymptotic distribution of the system lifetime. We show that under certain conditions the system lifetime normalized properly tends in distribution of the standard normal as the number of components increases to infinity.

Definition:

Let $X_1, X_2, \ldots$ be a sequence of independent random variables such that the distribution of X_k is F_k and $E[X_k] = 0$, $E|X_k^2| = \sigma_k^2$, $k = 1,2, \ldots$.

Let

$$S_n = X_1 + \ldots + X_n \ , \ s_n^2 = \sigma_1^2 + \ldots + \sigma_n^2 \ .$$

We say that the sequence $\{X_k\}$ satisfies Lindeberg Condition iff for each $\varepsilon > 0$:

$$\frac{1}{s_n^2} \sum_{k=1}^{n} \int_{|x|>\varepsilon s_n} x^2 dF_k(x) \to 0 \quad \text{as} \quad n \to \infty \ .$$

The following are well known theorems that we will use later on:

Theorem 2.8: (Lindeberg-Feller)

If Lindeberg Condition holds, the distribution of the normalized sums $\frac{S_n}{s_n}$ tends to the standard normal as $n \to \infty$.

We write $\frac{S_n}{s_n} \xrightarrow{\mathcal{D}} N(0,1)$ for convergence in distribution.

Theorem 2.9: (Slutzky)

If X_n tends in distribution to X and $\{A_n\}$, $\{B_n\}$ are two sequences of random variables converging in probability to a , b (constants) respectively, then $A_nX_n + B_n$ tends in distribution to $aX + b$.

Corollary 2.6: (Slutzky)

Suppose that a_n is a sequence of constants tending to ∞ , b is fixed number, and $a_n(X_n - b) \xrightarrow{\mathcal{D}} X$. Let g be a differentiable function, then $a_n(g(X_n) - g(b)) \xrightarrow{\mathcal{D}} g'(b)X$.

Theorem 2.10:

Let $X_1, X_2, \ldots$ be independent exponentially distributed random variables with mean $\theta_1, \theta_2, \ldots$ respectively and suppose that there exist $0 < L < U$ constants such that for each $n : L \leq \theta_n \leq U$, then $\{X_n - \theta_n\}$ satisfies Lindeberg Condition.

Proof:

The distribution of X_k ,

$$F_k(t) = 1 - e^{-t/\theta_k} \quad , \; t > 0$$

and hence for $A > 0$,

$$\int_{X>A} x^2 dF_k(x) = e^{-A/\theta_k}\left(A^2 + 2A\theta_k + 2\theta_k^2\right) .$$

Now since θ_k are uniformly bounded, check that for n large enough $\left(\text{e.g., } n > \left(\frac{U}{L}\right)^2\right)$ it suffices to consider only the right tail of the x's , that is,

$$\frac{1}{s_n^2}\sum_{k=1}^{n}\int_{|x-\theta_k|>\varepsilon s_n}(x-\theta_k)^2 dF_k(x) = \frac{1}{s_n^2}\sum_{k=1}^{n}\int_{x-\theta_k>\varepsilon s_n}(x-\theta_k)^2 dF_k(x)$$

but

$$\frac{1}{s_n^2}\sum_{k=1}^{n}\int_{x-\theta_k>\varepsilon s_n}(x-\theta_k)^2 dF_k(x)$$

$$\leq \frac{1}{s_n^2}\sum_{k=1}^{n}\int_{x>\varepsilon s_n} x^2 dF_k(x)$$

$$= \frac{1}{s_n^2}\sum_{k=1}^{n} e^{-\varepsilon s_n/\theta_k}\left(\varepsilon^2 s_n^2 + 2\varepsilon s_n\theta_k + 2\theta_k^2\right)$$

$$= \frac{1}{nL^2}\, ne^{-\varepsilon\sqrt{n}L/U}\left(\varepsilon^2 nU^2 + 2\varepsilon\sqrt{n}U^2 + 2U^2\right)$$

$$= \frac{n}{L^2}\, e^{-(\varepsilon L/U)\sqrt{n}}(\varepsilon^2 U^2 + 2\varepsilon U^2/\sqrt{n} + 2U^2/n)$$

which vanishes as $n \to \infty$.

Q.E.D.

Corollary 2.7:

Suppose that there exist $0 < A < B$ such that for any n , $A \leq \beta_n \leq B$, then:

$$\frac{\displaystyle\int_0^{T_{(N)}} L_s(u)du - \sum_{j=1}^{N} 1/\beta_j}{\left(\displaystyle\sum_{j=1}^{N} 1/\beta_j^2\right)^{1/2}} \xrightarrow{\mathcal{D}} N(0,1) \quad \text{as} \quad N \to \infty .$$

Corollary 2.8:

Let $\theta_k = \frac{1}{\beta_k}$. Suppose that $\theta_k = \theta + o\left(\frac{1}{\sqrt{k}}\right)$ as $k \to \infty$ and $\theta > 0$.

Then

$$\sqrt{N}\left(\frac{\int_0^{T_{(N)}} L_s(u)\,du}{N} - \theta\right) \xrightarrow{\mathcal{D}} N(0,\theta^2) \ .$$

Proof:

$$\sqrt{N}\left(\frac{\int_0^{T_{(N)}} L_s(u)\,du}{N} - \theta\right) = \sqrt{N}\left(\frac{L_s(u)\,du}{N}\right.$$

$$\left. - \frac{\sum_{k=1}^{n} \theta_k}{N}\right) + \sqrt{N}\left(\frac{\sum_{k=1}^{n} \theta_k}{N} - \theta\right) .$$

The first term equals

$$\frac{\sqrt{N}\,\dfrac{\int_0^{T_{(N)}} L_s(u)\,du - \sum_{k=1}^{N} \theta_k}{N}}{\left(\sum_{k=1}^{N} \theta_k^2/N\right)^{\frac{1}{2}}} \times \left(\frac{\sum_{k=1}^{N} \theta_k^2/N}{\theta}\right)^{\frac{1}{2}} \times \theta \ .$$

As $N \to \infty$ $\frac{1}{\theta}\left(\sum_{k=1}^{N} \theta_k^2/N\right)^{\frac{1}{2}} \to 1$ and the second term $\sqrt{N}\left(\sum_{k=1}^{N} \theta_k^2/N - \theta\right) \to 0$

and by Slutzky theorem, the result follows.

Q.E.D.

Example 1:

Suppose that $L_s(t) = L_s$ (constant) and $\theta_k = \theta + o\left(\frac{1}{\sqrt{k}}\right)$ as $k \to \infty$

where $\theta > 0$ the asymptotic distribution of the system lifetime $T_{(N)}$ is

$$\sqrt{N}\left(\frac{T_{(N)}}{N} - \frac{\theta}{L_s}\right) \xrightarrow{\mathcal{D}} N\left(0, (\theta/L_s)^2\right) \quad \text{as} \quad N \to \infty .$$

Example 2:

Suppose that $L_s(t) = \delta t$, $\delta > 0$, and the $\{\theta_k\}$ are as in Example 1. Then

$$\sqrt{N}\left(\frac{\delta}{2}\frac{T^2_{(N)}}{N} - \theta\right) \xrightarrow{\mathcal{D}} N(0,\theta^2) .$$

Use now Corollary 2.6 to show that:

$$T_{(N)} - \sqrt{\frac{2\theta}{\delta} N} \xrightarrow{\mathcal{D}} N(0,\theta/2\delta) \quad \text{as} \quad N \to \infty .$$

The interesting thing about this example is that for linear load program the asymptotic mean of $T_{(N)}$, $E[T_{(N)}] \approx \sqrt{N}$ whereas the asymptotic variance is constant.

2.4 More General Load Sharing Rules

A close study of Theorems 2.1 and 2.2 shows that the equal load sharing rule used throughout the previous sections is not essential for the derivation of most of the results so far. It seems that a system of components which obey the linear law breakdown rule is invariant, in some sense, with respect to the load sharing rule. In this section we assume a fairly general rule and prove that a modified version of Theorems 2.1 and 2.2 holds. Thus, using similar arguments as before we obtain similar results (i.e., IFR, Schur convexity, asymptotic distribution). In fact, we show that no matter how the total load is being distributed among the functioning components, the lifetime of the system remains stochastically the same.

Thus for instance, in analyzing the mechanical breakdown in bundles of fibers the "local load sharing" rule received some attention in the literature. The rule here is that every time a fiber fails, the adjacent fibers take on its load, we show that this system does not differ much from the equal load sharing system or from a system that allocates the total load to a single component at a time.

Again, by $T_1, T_2, \ldots, T_N$ we denote the lifetimes of components $1,2, \ldots, N$ respectively and $T_{(1)} \leq T_{(2)} \leq \cdots \leq T_{(N)}$ are their order statistics. $N(t)$ is the number of components that failed by time t $(t \geq 0)$. Let $\mathcal{H}_t$ denote the history of the system at time t, that is, $\mathcal{H}_t$ is the σ-field of events generated by the events of the form $\left\{T_{n_1} \leq t_1, T_{n_2} \leq t_2, \ldots, T_{n_k} \leq t_k, N(t) = k\right\}$ where $t_i \leq t$, $i = 1,2, \ldots, k$, $k \leq N$ and $\{n_1, \ldots, n_N\}$ is an arbitrary permutation of $\{1,2, \ldots, N\}$.

Now suppose that for each $t \geq 0$ there exist random variables $\alpha_1(t), \alpha_2(t), \ldots, \alpha_N(t)$ which are H_t measurable and satisfy the following:

(i) $\alpha_i(t) \geq 0$, $i = 1, \ldots, N$

(ii) $\alpha_i(t) = 0$ on $T_i < t$, $i = 1, \ldots, N$

(iii) $\sum_{i=1}^{N} \alpha_i(t) = 1$ on $T_{(N)} \leq t$

(iv) The sample path $\alpha_i(\cdot)$ is measurable.

The vector $\underline{\alpha}(t) = (\alpha_1(t), \ldots, \alpha_N(t))$ is the distribution program at time t , that is, if the load program of the system at t is $L_s(t)$, then component i ($i = 1,2, \ldots, N$) sustains at t load $\alpha_i(t)L_s(t)$ and hence the failure rate of component i at t is $\beta_i\alpha_i(t)L_s(t)$. Since $\alpha_i(t)$ is a random variable, we should interpret the failure rate in this case as:

$$\lim_{h \to 0^+} \frac{1}{h} P(T_i \in (t , t + h] \mid H_t) = \beta_i \alpha_i(t) L_s(t) \quad \text{w.p.1} .$$

It should be also clear that the atoms of H_t (i.e., the elementary events) are of the form: $\{T_{n_1} = t_1, \ldots, T_{n_k} = t_k , N(t) = k\}$ and since $\alpha_i(t)$ is H_t measurable, it is constant on these atoms. We denote the numerical value of $\alpha_i(t)$ on $\{T_{n_1} = t_1, \ldots, T_{n_k} = t_k , N(t) = k\}$ by $\alpha_i\left(t , T_{n_1} = t_1, \ldots, T_{n_k} = t_k , N(t) = k\right)$.

This is a generalization of the equal load sharing rule we treat in Section 2.1 where $\alpha_i(t)$ was there just $\frac{1}{N - N(t)}$ on $\{T_i > t\}$.

All other assumptions about the model remain the same.

We are ready now to state and prove lemmas and theorems similar to those in Section 2.1.

Lemma 2.6:

$$P(T_{(1)} > t) = \exp\left(-\int_0^t \sum_{j=1}^{N} \beta_j \alpha_j(u\ ,\ N(u) = 0)du\right) .$$

Proof:

Clearly for $\Delta > 0$:

$$P(T_{(1)} \in (t\ ,\ t+\Delta] \mid \mathcal{H}_t) = \begin{cases} \sum_{j=1}^{N} \beta_j \alpha_j(t\ ,\ N(t) = 0)\Delta + o(\Delta) \\ \text{as } \Delta \to 0\ , \text{ on } \{T_{(1)} > t\} \\ 0\ , \text{ otherwise.} \end{cases}$$

Thus the result follows by a similar way as in Lemma 2.1.

Q.E.D.

Lemma 2.7:

Let $0 \le t_1 \le t_2 \le \dots \le t_k \le t$, $k < N$, and let $\{n_1, \dots, n_N\}$ be an arbitrary permutation of $\{1,2, \dots, N\}$, then for $x > 0$

$$P\left(T_{(k+1)} > t + x \mid T_{n_1} = t_1, T_{n_2} = t_2, \dots, T_{n_k} = t_k\ ,\ N(t) = k\right)$$

$$= \exp\left(-\int_t^{t+x} \sum_{i=1}^{N} \beta_i \alpha_i\left(u\ ,\ T_{n_1} = t_1, \dots, T_{n_k} = t_k\ ,\ N(u) = k\right) L_s(u)du\right) .$$

Proof:

Similar to Lemma 2.2.

Q.E.D.

Lemma 2.8:

$\{t_i\}$, $\{n_i\}$, k , as in Lemma 2.6 and let $\underline{dt}$ denote the infinitesimal interval $(t , t + dt]$, then for $x > 0$,

$$P\left(T_{n_k} \in \underline{dt} , T_{(k+1)} > t + x \mid T_{n_1} = t_1, \ldots, T_{n_{k-1}} = t_{k-1} , N(t) = k - 1\right)$$

$$= \left[\alpha_{n_k}\left(t , T_{n_1} = t_1, \ldots, T_{n_{k-1}} = t_{k-1} , N(t) = k - 1\right) L_s(t) \times \exp\left(- \int_t^{t+x} \sum_{j=1}^{N} \beta_j \alpha_j\left(u , T_{n_1} = t_1, \ldots, T_{n_k} = t_k , N(u) = k\right) du\right)\right] dt .$$

Proof:

Similar to Lemma 2.3.

Q.E.D.

The following generalizes Theorem 2.1.

Theorem 2.11:

Consider the simplex $0 < t_1 \leq t_2 \leq \ldots \leq t_N$ and $\{n_j\}$ is as in Lemma 2.6 then:

$$P\left(T_{n_1} \in \underline{dt}_1, \ldots, T_{n_N} \in \underline{dt}_n\right) = \left[\beta_{n_1} L_s(t_1)\alpha_{n_1}(t \ , N(t_1) = 0)dt_1\right]$$

$$\times \left[\prod_{j=2}^{N} \beta_{n_j} L_s(t_j)\alpha_{n_j}\left(t_j \ , T_{n_1} = t_1, \ldots, T_{n_{j-1}} = t_{j-1} \ ,\right.\right.$$

$$\left.\left. N(t_j) = j - 1\right)dt_j\right]$$

$$\times \exp - \left\{\int_0^{t_1} \sum_{j=1}^{N} \beta_j\alpha_j(u \ , N(u) = 0)L_s(u)du\right.$$

$$+ \int_{t_1}^{t_2} \sum_{j=1}^{N} \beta_j\alpha_j\left(u \ , T_{n_1} = t_1 \ , N(u) = 1\right)L_s(u)du + \ldots +$$

$$\left.\int_{t_{N-1}}^{t_N} \sum_{j=1}^{N} \beta_j\alpha_j\left(u \ , T_{n_1} = t_1, \ldots, T_{n_{N-1}} = t_{N-1} \ , N(u) = N - 1\right)L_s(u)du\right\} .$$

Proof:

Similar to Theorem 2.1 using here Lemmas 2.5, 2.6, and 2.7.

Q.E.D.

As in Section 2.1, we define here the vector $(W_1, W_2, \ldots, W_N)$, where

$$W_i = \int_0^{T_i} \alpha_i(u)L_s(u)du \ , \ i = 1,2, \ldots, N \ .$$

Thus, the W's are the total loads sustained by components $1,2, \ldots, N$. The following generalizes Theorem 2.2:

Theorem 2.12:

The random variables $W_1, \ldots, W_N$ are independent exponentially distributed with rates $\beta_1, \beta_2, \ldots, \beta_N$ respectively.

Proof:

Very similar to the proof of Theorem 2.2.

Here is the sketch of it. Consider the simplex $0 < t_1 \leq t_2 \leq \cdots \leq t_N$ and $\{n_1, \ldots, n_N\}$ a permutation of $1, \ldots, N$, then on $\left\{T_{n_1} = t_1, \ldots, T_{n_N} = t_N\right\}$ the vector $(W_1, \ldots, W_N)$ assumes the values

$$W_{n_1} = \alpha_{n_1}(u\ , N(u) = 0)L_s(u)du = w_{n_1}$$

$$W_{n_2} = \int_0^{t_1} \alpha_{n_2}(u\ , N(u) = 0)L_s(u)du$$

$$+ \int_{t_1}^{t_2} \alpha_{n_2}\left((u)\ , T_{n_1} = t_1\ , N(u) = 1\right)L_s(u)du = w_{n_2}$$

$$\vdots \quad \text{etc.}$$

Hence the joint density function of $\underline{W}$ at the point $\underline{w}$ is just

$$f_{W_{n_1}, \ldots, W_{n_N}}\left(w_{n_1}, \ldots, w_{n_N}\right) =$$

$$f_{T_{n_1}, \ldots, T_{n_N}}(t_1, \ldots, t_N)\left|\left|\frac{\partial(w_1, \ldots, w_n)}{\partial(t_1, \ldots, t_N)}\right|\right|^{-1} .$$

Check that

$$\left|\left|\frac{\partial w}{\partial t}\right|\right| = \prod_{j=1}^{N} L_s(t_j)\alpha_{n_j}\left(t_j\ , T_{n_1} = t_1, \ldots, T_{n_{j-1}} = t_{j-1}\ , N(t_j) = j - 1\right) ,$$

where the first term in the product should be interpreted as $L_s(t_1)\alpha_{n_1}(t_1, N(t_1) = 0)$. Combine this with Theorem 2.11 to obtain

$$f_{\underline{W}}(w_1, \ldots, w_N) = \prod_{j=1}^{N} \beta_j e^{-\beta_j w_j} , \quad w_j \geq 0 .$$

The above theorem provides us with most of the results that we obtained for the equal load sharing rule. For completeness we state them here without a proof. Throughout the rest of this section fix $L_s(\cdot)$ and let $\{\underline{\alpha}(t) : t \geq 0\}$ be fixed and satisfying conditions (2.3). Note that under these conditions

$$\int_0^{T_{(N)}} L_s(u)du = \sum_{i=1}^{N} W_i .$$

Theorem 2.13:

If $L_s(\cdot)$ is nondecreasing, then the system's lifetimes is IFR.

Theorem 2.14:

Fix $t > 0$, then $P(T_{(N)} > t)$ is Schur convex function in $(\beta_1, \ldots, \beta_N)$.

Corollary 2.9:

Let $\beta = \frac{1}{N} \sum_{i=1}^{N} \beta_i$, then for any $t > 0$

$$P(T_{(N)} > t) \geq e^{-\beta \int_0^t L_s(u)du} \frac{\sum_{i=0}^{N-1} \beta^i \left(\int_0^t L_s(u)du\right)^i}{i!} .$$

Theorem 2.15:

Suppose there exist $0 < A < B$ (constants) such that for any $n \geq 1$ $A \leq \beta_n \leq B$, then

$$\frac{\int_0^{T_{(N)}} L_s(u)du - \sum_{j=1}^{N} 1/\beta_j}{\left(\sum_{j=1}^{N} 1/\beta_j^2\right)^{\frac{1}{2}}} \xrightarrow{\mathcal{D}} N(0,1) \quad \text{as} \quad N \to \infty .$$

Theorem 2.16:

Let $\theta_k = 1/\beta_k$, $k = 1,2, \ldots, N$. Suppose that

$$\theta_k = \theta + o\left(\frac{1}{\sqrt{k}}\right) \quad \text{as} \quad k \to \infty \quad \text{and} \quad \theta > 0$$

then

$$\sqrt{N}\left(\frac{\int_0^{T_{(N)}} L_s(u)du}{N} - \theta\right) \xrightarrow{\mathcal{D}} N(0,\theta^2) \quad \text{as} \quad N \to \theta .$$

CHAPTER 3

GENERAL BREAKDOWN RULE

In this chapter we generalize the assumption about the breakdown rule. Whereas, in the previous chapter we consider only a linear-power breakdown rule, here we allow it to be quite general. We basically adapt the model which was treated by Coleman (1957, 1958) and more recently by Phoenix (1978). In all previous studies, the models assumed parallel system of identical components. Here we generalize it and study systems of nonidentical components.

We again consider an N-component parallel system which is subjected to a nonnegative time dependent load program $L_s(\cdot)$. Thus the nominal load per component is $L_s(t)/N$. Throughout this chapter at any time $t > 0$, the surviving components are assumed to share the load equally. If we denote by $N(t)$ the number of components that failed prior to time t , the actual load carried by each of the $N - N(t)$ surviving components at t is $L_s(t)/N - N(t)$. Hence the actual component load program is a stochastic process which we denote by

$$L(t) = \begin{cases} L_s(t)/N - N(t) & \text{on} \quad N(t) < N \\ L_s(t) & \text{on} \quad N(t) = N \ . \end{cases}$$

Each component, therefore, is being subjected to $L(t)$ up to its time of failure after which its load is being distributed equally among the rest of the surviving components.

Assumptions about the individual components: Component i $(i = 1, 2, \ldots, N)$ when it is being subjected to a known load program $l(\cdot)$

has a random failure time T_i which is distributed according to $F_i(t \mid l(\cdot))$, where

$$F_i(t \mid l(\cdot)) = 1 - e^{-\psi_i\left(\int_0^t K(l(u))du\right)}, \quad t > 0 \tag{3.1}$$

where ψ_i, $i = 1, \ldots, N$, and K satisfy:

(i) for each i, ψ_i is increasing, continuous, $\psi_i(0) = 0$, $\psi_i(\infty) = \infty$.

(ii) K is positive increasing, unbounded (i.e., $K(\infty) = \infty$ and has continuous derivative $K'(\cdot)$.

If we denote by $G_i(x) = 1 - e^{-\psi_i(x)}$, $i = 1, \ldots, N$, then G_i is a distribution on the positive line and $F_i(t \mid l) = G_i\left(\int_0^t K(l(u))du\right)$. If ψ_i is also differentiable, then under a known load program $l(\cdot)$, component i possesses a failure rate function:

$$\lambda_i(t \mid l) = \psi_i'\left(\int_0^t K(l(u))du\right)K(l(t)). \tag{3.2}$$

Note that in this case the failure rate function contains two factors $\psi_i'\left(\int_0^t K(l(u))du\right)$ which characterizes the effect of the load history and $K(l(t))$ which characterizes the effect of the present load. As in the previous chapter we should emphasize that the components life-times are stochastically dependent, but only through their failure rate functions (see introduction to Chapter 2).

Our approach is similar to the one used by Phoenix (1978). We will show that the system lifetime (under certain load programs) can be expressed as a linear combination of order statistics of independent random variables with known distribution functions. To obtain the asymptotic distribution of the system lifetime (as the number of components increase to infinity) we use Shorack (1973), Stigler (1974), and some more recent result of Wesley (1977) which also points out an error in the original version of Stigler. The error in Stigler's paper leads to an unjustified statement in Phoenix's paper which we will point out and give a correct version.

3.1 Preliminaries

In this section we review some of the results obtained in Shorack (1973) and Stigler (1974). Our interest in these papers derives from the fact that our result leans heavily on them. Their results are quite general which is more than we need, thus we shall bring only a modified version which fits our case.

Let $X_{n,1}, X_{n,2}, \ldots, X_{n,n}$ be a nonnegative triangular array of row independent random variables having continuous distribution fucntions $F_{n,1}, F_{n,2}, \ldots, F_{n,n}$ and $\tilde{F}_n = \frac{1}{n} \sum_{i=1}^{n} F_{n,i}$ and let $X_{(n,1)} \le X_{(n,2)} \le \ldots \le X_{(n,n)}$ be their order statistics. Shorack and Stigler were interested in the asymptotic (as $n \to \infty$) distribution of:

$$S_n = \frac{1}{n} \sum_{i=1}^{n} C_{n,i} X_{(n,i)}$$

where $C_{n,i}$ are known constants.

If we define J_n on $[0,1]$ by

$$J_n(t) = C_{n,i} \,, \quad \frac{i-1}{n} < i \leq \frac{i}{n}$$

$$J_n(0) = C_{n,1} \,,$$

then

$$S_n = \frac{1}{n} \sum_{i=1}^{n} J_n(i/n) X_{(n,i)} \,.$$

Now suppose the following conditions hold:

1. Let J be continuous on $(0,1)$ and suppose $J_n \to J$ uniformly on $[\theta \,, 1-\theta]$, as $n \to \infty$ for each $\theta > 0$.
2. For some F^{-1}, $\tilde{F}_n^{-1}(t) \to F^{-1}(t)$ for each continuity point t of F^{-1}.
3. J, J_n are uniformly bounded on $(0,1)$.
4. $F^{-1}(t) \leq D(t)$, $\tilde{F}_n^{-1}(t) \leq D(t)$ for $0 < t < 1$, where $D(t) = M[t(1-t)]^{-\frac{1}{2}+\delta}$ for some $\delta > 0$.

Then:

<u>Theorem 3.1</u>:

If conditions 1, 2, 3, 4 hold, then

$$\sqrt{n}(S_n - \mu_n) \xrightarrow{\mathcal{L}} N(0,\sigma^2)$$

where

$$\mu_n = \int_0^1 J_n(u) F_n^{-1}(u)\,du$$

$$\sigma^2 = \int_0^1\int_0^1 J(s)J(t)K(t,s)\,dF^{-1}(t)\,dF^{-1}(s)$$

and

$$K(s,t) = s \wedge t - \lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} F_{n,i}\left(F_n^{-1}(t)\right) F_{n,i}\left(F_n^{-1}(s)\right) .$$

<u>Proof</u>:

Shorack (1973), Theorem 3.1.

σ^2 can be rewritten as

$$\sigma^2 = \int_0^\infty\int_0^\infty J(F(x))J(F(y))\Gamma(x,y)\,dx\,dy$$

where

$$\Gamma(x,y) = \lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} [F_{n,i}(x \wedge y) - F_{n,i}(x)F_{n,i}(y)] .$$

Q.E.D.

<u>Theorem 3.2</u>:

If the above conditions hold, then

$$ES_n \to \int_0^1 J(u)F^{-1}(u)\,du \quad \text{as} \quad n \to \infty .$$

3.2 The Result: (The Asymptotic Distribution of $T_{(N)}$ as $N \to \infty$)

We denote by $T_1, T_2, \ldots, T_N$ the lifetimes of components 1,2, ..., N respectively and by $T_{(1)}, T_{(2)}, \ldots, T_{(N)}$ their order statistics. We assume that the system is being subjected to a nonnegative load program $L_s(\cdot)$ and that each component obeys the breakdown rule given in 3.2. (That is, we assume that ψ_i', $i = 1, \ldots, N$ exists and the failure rate function under a known load program of an individual component is given by 3.2.) To begin with, we obtain the joint density function of $T_1, \ldots, T_N$ under $L_s(\cdot)$ using an approach similar to the one in the previous chapter. The assumptions of the model described before lead clearly to the following facts:

(F1) For each i $(i = 1, \ldots, N)$,

$$P_{L_s}(T_i \in \underline{dt} \mid T_{(1)} > t) = \psi_i\left(\int_0^t K\left(\frac{L_s(u)}{N}\right)du\right)K\left(\frac{L_s(t)}{N}\right)dt \; .$$

(F2) If $0 = t_0 < t_1 < t_2 < \ldots < t_k < t$ $k < j \le N$ and $\{n_1, \ldots, n_N\}$ is an arbitrary permutation of $\{1, \ldots, N\}$, then

$$P_{L_s}\left(T_{n_j} \in \underline{dt} \mid T_{n_1} = t_1, \ldots, T_{n_k} = t_k \; , \; T_{(k+1)} > t\right) =$$

$$\psi_{n_j}\left(\int_0^t K(L(u,t_1, \ldots, t_k))du\right)K\left(\frac{L_s(t)}{N - k}\right)dt$$

where

$$L(u,t_1, \ldots, t_k) = \begin{cases} \dfrac{L_s(t)}{N-j}, & t_j \le t < t_{j+1}, \\ & j = 0,1, \ldots, k-1 \\ \dfrac{L_s(t)}{N-k}, & t \ge t_k \end{cases}$$

and dt is the infinitesimal interval $(t , t + dt]$.

The following lemmas are a direct consequence of (F1), (F2) and their proofs are similar to Lemmas (2.1), (2.2), (2.3) which we omit here.

Lemma 3.1:

$$P_{L_s}(T_{(1)} > t) = e^{-\sum_{i=1}^{n} \psi_i\left(\int_0^t K\left(\frac{L_s(u)}{N}\right)du\right)} .$$

Lemma 3.2:

Let $0 = t_0 \le t_1 \le t_2 \le \ldots \le t_k < t \quad k < N$ and let $n_1, \ldots, n_N$ be an arbitrary permutation of $1,2, \ldots, N$, then for $x > 0$ we have:

$$P_{L_s}\left(T_{(k+1)} > t + x \mid T_{n_1} = t_1, \ldots, T_{n_k} = t_k , T_{(k+1)} > t\right) =$$

$$e^{-\sum_{j=k+1}^{N}\left[\psi_{n_j}\left(\int_0^{t+x} K(L(u,t_1, \ldots, t_k))du\right) - \psi_{n_j}\left(\int_0^{t} K(L(u,t_1, \ldots, t_k))du\right)\right]}$$

where $L(t,t_1, \ldots, t_k)$ is defined in (F2).

Lemma 3.3

$\{t_i\}$, $\{n_i\}$ as in Lemma 3.2 then for $x > 0$

$$P_{L_s}\left(T_{n_k} \in \underline{dt} \, , \, T_{(k+1)} > t + x \mid T_{n_1} = t_1, \ldots, T_{n_{k-1}} = t_{k-1} \, , \, T_{(k)} > t\right)$$

$$= \psi'_{n_k}\left(\int_0^t K(L(u,t_1, \ldots, t_{k-1}))du\right) K\left(\frac{L_s(t)}{N-k+1}\right)dt$$

$$e^{-\sum_{j=k+1}^{N}\left[\psi_{n_j}\left(\int_0^{t+x} K(L(u,t_1, \ldots, t_k))du\right) - \psi_{n_j}\left(\int_0^{t} K(L(u,t_1, \ldots, t_k))du\right)\right]} .$$

We are now ready to obtain the joint density function of $T_1, \ldots, T_N$.

Theorem 3.3

Let $\{n_i\}$, $\{t_i\}$ be as in Lemma 2, then

$$P_{L_s}\left(T_{n_i} \in \underline{dt}_i \quad i = 1, \ldots, N\right) =$$

$$\prod_{i=1}^{N}\left[\psi'_{n_i}\left(\int_0^{t_i} K(L(u,t_1, \ldots, t_N))du\right) K(L(t_i,t_1, \ldots, t_N))dt_i \times e^{-\psi_{n_i}\left(\int_0^{t_i} K(L(u,t_1, \ldots, t_N))du\right)}\right]$$

where

$$L(u,t_1, \ldots, t_N) = \begin{cases} \dfrac{L_s(u)}{N-i} & t_i \leq u < t_{i+1} \quad i = 0,1, \ldots, N-1 \\ L_s(u) & u \geq t_N \, . \end{cases}$$

Proof:

Combine Lemmas 1, 2, 3 in a similar way as in Theorem 2.1.

Q.E.D.

Next we generalize Theorem 2.2 to this model:

Consider $L(t, T_{(1)}, T_{(2)}, \ldots, T_{(N)})$ where L is as defined above and $\{T_{(j)}\}$ is the order statistics and defined by:

$$(3.3) \qquad Q_i = \int_0^{T_i} K(L(u, T_{(1)}, \ldots, T_{(N)}))du$$

then the following theorem is a general version of Theorem 2.2:

Theorem 3.4:

The random variables $Q_1, \ldots, Q_N$ are independent and distributed according to $G_1, \ldots, G_N$ respectively, where

$$G_i(x) = 1 - e^{-\psi_i(x)} \qquad x \geq 0 .$$

Proof:

We prove it by obtaining the joint density of $Q_1, \ldots, Q_N$. Since $K(\cdot)$ is (by assumption) increasing and the equal load sharing is applied, then under $L_s(\cdot) > 0$ the mapping defined in (3.3) is one to one and hence the joint density of $Q_1, \ldots, Q_N$ can be expressed in terms of the joint density function of $T_1, T_2, \ldots, T_N$. Let $\{n_1, \ldots, n_N\}$ be an arbitrary permutation of $\{1, \ldots, N\}$, if $f_{Q_{n_1}, \ldots, Q_{n_N}}$ and

$f_{T_{n_1}, \ldots, T_{n_N}}$ are the joint densities of the Q's and T's respectively, then $f_{Q_{n_1}, \ldots, Q_{n_N}} = f_{T_{n_1}, \ldots, T_{n_N}} ||J||^{-1}$,

where $||J|| = \left|\left|\dfrac{\partial(q_1, \ldots, q_N)}{\partial(t_1, \ldots, t_N)}\right|\right|$. Now consider $0 < q_1 < q_2 < \ldots < q_N$ to find $f_{Q_{n_1}, \ldots, Q_{n_N}}(q_1, \ldots, q_N)$ let $t_1 < t_2 < \ldots < t_N$ be such that,

$$q_1 = \int_0^{t_1} K\left(\frac{L_s(u)}{N}\right) du$$

$$q_2 = q_1 + \int_{t_1}^{t_2} K\left(\frac{L_s(u)}{N-1}\right) du$$

$$\vdots$$

$$q_N = q_{N-1} + \int_{t_{N-1}}^{t_N} K(L_s(u))\, du \,.$$

It is easy to see that $\{t_i\}$ are uniquely defined and furthermore,

$$J = \begin{pmatrix} K\left(\frac{L_s(t_1)}{N}\right) & & & \bigcirc \\ \vdots & K\left(\frac{L_s(t_2)}{N-1}\right) & & \\ \vdots & & \ddots & \\ \cdots & \cdots & \cdots & K\left(\frac{L_s(t_N)}{1}\right) \end{pmatrix}$$

Thus:

$$f_{Q_{n_1}, \ldots, Q_{n_N}}(q_1, \ldots, q_N) = f_{T_{n_1}, \ldots, T_{n_N}}(t_1, \ldots, t_N)||J||^{-1}$$

$$= \prod_{i=1}^{N} \psi'_{n_i}(q_i)e^{-\psi_{n_i}(q_i)} \quad , \; 0 < q_1 < q_2 < \ldots < q_N \; .$$

Since this argument holds for any arbitrary $\{n_i\}$, the proof is complete.

Q.E.D.

Remark:

The above theorem holds even when the ψ_i's are not differentiable. This theorem, as we pointed out before, is the adaptation of Theorem 2.2 to the more general model and its importance is that it enables us to express $T_{(N)}$ -- system lifetimes as a weighted sum of $Q_{(1)}, \ldots, Q_{(N)}$ -- the order statistics of the Q's , then use Stigler's results to derive the asymptotic distribution of $T_{(N)}$ as $N \to \infty$. The same approach was used by Phoenix (1978) to derive the asymptotic results for the identical-component system.

With the function $K(\cdot)$ mentioned in (3.2) we associate the function

$$\phi(x,\lambda) = \begin{cases} \dfrac{1}{K\left(\dfrac{\lambda}{1-x}\right)} & , \; 0 \leq x < 1 \; , \; \lambda > 0 \\ 0 & , \; x = 1 \end{cases} \tag{3.4}$$

From the assumptions about $K(\cdot)$, it follows that $\frac{\partial}{\partial x}\phi(x,\lambda) \equiv \phi'(x,\lambda)$ is decreasing and continuous on $[0,1]$.

To illustrate the method we will first consider the case in which the system load program $L_s(t) = L_s$ (constant). Thus if $T_1, \ldots, T_N$ and $Q_1, \ldots, Q_N$ are as defined in (3.3) and if $T_{(1)}, T_{(2)}, \ldots, T_{(N)}$, $Q_{(1)}, \ldots, Q_{(N)}$ are their order statistics then under $L_s(t) = NL$ (constant) we have:

$$T_{(1)} = \phi(0,L)Q_{(1)}$$

$$T_{(i+1)} - T_{(i)} = \phi\left(\frac{i}{N},L\right)\left[Q_{(i+1)} - Q_{(i)}\right] \quad i = 1,2, \ldots, N-1$$

thus

$$T_{(N)} = \sum_{i=1}^{N} \left[\phi\left(\frac{i-1}{N},L\right) - \phi\left(\frac{i}{N},L\right)\right] Q_{(i)} \ .$$

We define

$$\text{(3.5)} \qquad \begin{aligned} J_N(x) &= N\left\{\phi\left(\frac{i-1}{N},L\right) - \phi\left(\frac{i}{N},L\right)\right\} \quad \frac{i-1}{N} < x \le \frac{i}{N} \\ J_N(0) &= N\left\{\phi(0,L) - \phi\left(\frac{1}{N},L\right)\right\} \ . \end{aligned}$$

Then

$$T_{(N)} = \frac{1}{N} \sum_{i=1}^{N} J_N\left(\frac{i}{N}\right)Q_{(i)}$$

and furthermore:

<u>Lemma 3.4</u>:

$J_N \to -\phi'(\cdot,L)$ uniformly in $[0,1]$ as $N \to \infty$.

Proof:

By assumption, ϕ' is continuous and hence uniformly continuous in $[0,1]$. By the mean value theorem $\exists\ t_i \in \left(\frac{i-1}{N},\frac{i}{N}\right)$ such that $n\left\{\phi\left(\frac{i-1}{n},L\right) - \phi\left(\frac{i}{n},L\right)\right\} = -\phi'(t_i,L)$, so for each $x \in [0,1]\ \exists\ t$ with $|t - x| < 1/n$ such that $|-\phi'(x,L) - J_N(x)| = |-\phi'(x,L) + \phi'(t,L)|$ which by the uniform continuity of $-\phi'$ is uniformly small as $n \to \infty$.

Q.E.D.

In fact $T_{(N)}$ can be rewritten in the following way

$$T_{(N)} = \frac{1}{N}\sum_{i=1}^{N} -\phi'(t_{N,i},L)Q_{(i)}$$

for some $\{t_{N,i}\}$

$$t_{N,i} \in \left(\frac{i-1}{N},\frac{i}{N}\right) .$$

Corollary 3.1:

$\{J_N\}$ are uniformly bounded on $[0,1]$.

Theorem 3.5:

Suppose that the following hold:

(i) For some G^{-1} , $\tilde{G}_n^{-1}(t) \to G^{-1}(t)$ for each continuity point of G^{-1} , where

$$\tilde{G}_n = \frac{1}{n}\sum_{i=1}^{n} G_i$$

and $\{G_i\}$ are from Theorem 3.2.

(ii) $\tilde{G}_n^{-1}$, G^{-1} are bounded above by $M[t(1-t)]^{-\frac{1}{2}+\delta}$ for some $\delta > 0$, $M > 0$.

Then under the constant load program $L_s = NL$

$$\sqrt{N}(T_{(N)} - \mu_N) \xrightarrow{\mathcal{D}} N(0,\sigma^2) \quad \text{as} \quad N \to \infty$$

where

$$\mu_N = \int_0^1 J_N(u)\tilde{G}_n^{-1}(u)du$$

and

$$\sigma^2 = \int_0^1\int_0^1 \phi'(s,L)\phi'(t,L)K(s,t)dG^{-1}(t)dG^{-1}(s)$$

where

$$K(s,t) = s \wedge t - \lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} G_i\left(\tilde{G}_n^{-1}(t)\right) G_i\left(\tilde{G}_n^{-1}(s)\right) .$$

Furthermore the asymptotic mean of $T_{(N)}$ is given by

$$\mu = \int_0^1 -\phi'(u,L)G^{-1}(u)du .$$

<u>Proof</u>:

All conditions of Theorem 3.1 hold.

Q.E.D.

In his paper, Phoenix obtained similar results for the identical components case. However, he went on to conclude that $\sqrt{n}(T_{(n)} = \mu) \xrightarrow{\mathcal{D}} N(0,\sigma^2)$. Unfortunately, this is not a direct consequence of the above theorem, unless one shows that $ET_{(n)} = \mu + o\left(\frac{1}{\sqrt{n}}\right)$ as $n \to \infty$. Stigler (1974) addressed this problem (see Theorem 4 there) but as it turned out his argument was not complete (see for instance Stigler (1979) or Wesley (1977)). Thus as far as we can tell, Theorem 2 and Corollary 1 in Phoenix (1978) still need to be justified. One possible solution was provided by Shorack (1972) (see Example 1), which can be stated as follows:

For the identical component system (i.e., $G_i = G \ \forall \ i$) under a constant load program $L_s = NL$.

<u>Theorem 3.6</u>:

(i) For some $r > 0$, $\int x^r dG(x) < \infty$.

(ii) For some $\delta > 0$, $M > 0$, $|\phi'(x,L)| \leq M[t(1-t)]^{-\frac{1}{2}+\frac{1}{r}+\delta}$.

(iii) $\phi''(\cdot,L)$ exists and continuous on $(0,1)$ and

$$|\phi''(x,L)| \leq M[t(1-t)]^{-\frac{3}{2}+\frac{1}{r}+\delta}$$

then

$$\sqrt{N}(T_{(N)} - \mu) \xrightarrow{\mathcal{D}} N(0,\sigma^2) \quad \text{as} \quad N \to \infty$$

where μ , σ^2 are as in Theorem 3.2.

Proof:

See Example 1, Shorack (1972).

Q.E.D.

Next we show how to derive similar results for a more arbitrary load program. To avoid complicated expressions, we restrict ourselves to the power-law breakdown rule (i.e., $K(t) = (Kt)^\rho$, $\rho > 1$ and suppose $L_s(t) = Nl(t)$, thus $l(t)$ is the nominal load per component. By the definitions of $\{T_{(i)}\}$, $\{Q_{(i)}\}$ we have

$$Q_{(1)} = \int_0^{T_{(1)}} [Kl(u)]^\rho du$$

$$Q_{(i+1)} - Q_{(i)} = \int_{T_{(i)}}^{T_{(i+1)}} \left(\frac{Kl(u)}{1 - i/N}\right)^\rho du \ , \ i = 1, \ldots, N - 1$$

thus

$$\int_0^{T_{(1)}} (l(u))^\rho du = K^{-\rho} Q_{(1)}$$

$$\int_{T_{(i)}}^{T_{(i+1)}} (l(u))^\rho du = \left(\frac{1 - 1/N}{K}\right)^\rho [Q_{(i+1)} - Q_{(i)}]$$

and

$$\int_0^{T_{(N)}} (l(u))^\rho du = \sum_{i=1}^{N} \left[\left(\frac{1 - \frac{i - 1}{N}}{K}\right)^\rho - \left(\frac{1 - \frac{i}{N}}{K}\right)^\rho\right] Q_{(i)} \ .$$

Define: $\phi(x,k) = \left(\frac{1-x}{k}\right)^{\rho}$.

Theorem 3.7:

If the conditions of Theorem 3.2 hold, then

$$\sqrt{N}\left[\int_0^{T_{(N)}} (l(u))^{\rho} du - \mu_N\right] \xrightarrow{\mathcal{D}} N(0,\sigma^2) \quad \text{as} \quad N \to \infty$$

where μ_N , σ^2 are as defined in Theorem 3.2. Suppose $l(\cdot) > 0$, and if

$$h(t) = \int_0^t [l(u)]^{\rho} du \ , \ t \geq 0$$

then h^{-1} is well defined and

Corollary 3.2:

$$\sqrt{N}\left(T_{(N)} - h^{-1}(\mu_N)\right) \xrightarrow{\mathcal{D}} N\left(0,\left(\frac{d}{dx} h^{-1}(\mu)\right)^2 \sigma^2\right) \quad \text{as} \quad N \to \infty$$

where μ , μ_N , σ^2 as in Theorem 3.2.

Proof:

The above is a special case of Slutsky theorem.

Q.E.D.

REFERENCES

[1] Birnbaum, Z. W. and S. C. Saunders, "A Statistical Model for Life Length of Materials," J. Amer. Stat. Assoc., Vol. 53, pp. 151-159, (1958).

[2] Coleman, B. D., "A Stochastic Process Model for Mechanical Breakdown," Trans. Soc. Rheol. 1, pp. 153-168, (1957).

[3] Coleman, B. D., "Time Dependent of Mechanical Breakdown in Bundles of Fibers III: The Power Law Breakdown Rule," Trans. Soc. Rheol. 2, pp. 195-218, (1958).

[4] Coleman, B. D., "Statistics and Time Dependence of Mechanical Breakdown in Fibers," J. Appl. Phys., Vol. 29, pp. 968-983, (1958).

[5] Esary, J. D., F. Proschan and D. W. Walkup, "Association of Random Variables with Applications," Ann. Math. Stat., Vol. 38, pp. 1466-1474, (1967).

[6] Feller, W., AN INTRODUCTION TO PROBABILITY THEORY AND ITS APPLICATIONS, Vol. II, 2nd Ed., Wiley, New York, (1971).

[7] Hardy, G. H., J. E. Littlewood and G. Polya, INEQUALITIES, Cambridge University Press, (1934).

[8] Ostrowski, A., "Sur Quelques Applications des Founctions Convex et Concaves an Sens de I. Schur," J. Math. Pures Appl., Vol. 31, pp. 253-292, (1964).

[9] Phoenix, L. S., "The Asymptotic Time to Failure of a Mechanical System of Parallel Members," SIAM, J. Appl. Math., Vol. 34, No. 2, pp. 227-247, (1978).

[10] Proschan, F., "Applications of Majorization and Schur Functions in Reliability and Lifetesting, RELIABILITY AND FAULT TREE ANALYSIS, SIAM, (1975).

[11] Proschan, F. and J. Sethuraman, "Stochastic Comparison of Sums, Using Majorization," Technical Report M308, Department of Statistics, Florida State University, (1974).

[12] Snyder, D. L., RANDOM POINT PROCESSES, Wiley, New York, (1975).

[13] Shorack, G. R., "Functions of Order Statistics," Ann. Math. Stat., Vol. 43, pp. 412-427, (1972).

[14] Shorack, G. R., "Convergence of Reduced Emperical and Quntile Processes with Applications to Functions of Order Statistics in the Non-i.i.d. Case," Ann. Math. Stat., Vol. 1, pp. 146-152, (1973).

[15] Stigler, S. M., "Linear Functions of Order Statistics with Smooth Weight Functions," Ann. Stat., Vol. 1, pp. 676-693, (1974).

[16] Stigler, S. M., "Correction to Linear Functions of Order Statistics with Smooth Weight Functions," Ann. Stat., Vol. 7, p. 466, (1979).

[17] Wesley, R. A., "Contributions to the Theory of Robust Regression," Technical Report No. 2, Department of Statistics, Stanford University, (December 1977).

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